Recursive definition of the lattice of Moore families

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Abstract A collection of sets on a ground set S_n ($S_n = \{1, 2, ..., n\}$) closed under intersection and containing S_n is known as a Moore family. The set of Moore families for a fixed n is a lattice denoted $\mathbb{M}_n$. In this paper we provide a recursive definition of $\mathbb{M}_n$. This alternative definition puts highlight some new structural properties of the lattice of Moore families.

1 Introduction

The concept of a collection of sets closed under intersection appears with different names depending on the scientific fields. The name *Moore family* was first used by Birkhoff in [3] referring to E.H. Moore's research. But, very frequently, such a collection on a ground set S_n ($S_n = \{1, 2, ..., n\}$) is called closure system. This concept is applied to numerous fields in pure or applied mathematics and computer science. For instance Cohn, Sierksma and van de Vel have used it in the framework of algebra and topology ([7, 14, 15]) while Birkhoff, Davey and Priestley focused on order and lattice theory ([2, 10]). Formally a closure operator is an extensive, isotone and idempotent function on 2^{S_n} (the set of all subsets of S_n), and a closure system is then the set of its fixed points. In particular it is well-known that any closure system is a complete lattice. In 1937 Birkhoff ([2]) gave a compact representation of *quasi-ordinal spaces* (in other words of collections of sets closed under intersection and union and so which are distributive lattices). More recently the notion of closure system appears as a significant concept in computer science with research in relational databases ([9]), data analysis and formal concept analysis ([12, 1, 13]). More precisely, Ganter and Wille defined a mathematical framework for classification, and Barbut defined and used Galois lattices about questions raised in Guttman scales analysis ([1]). Meanwhile, in 1985, equivalent collections of sets were called *knowledge spaces* by Doignon and Falmagne ([8]) to study possible states of knowledge of a student.

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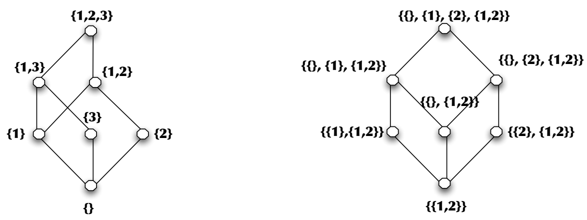
An important fact is that the collection of Moore families on S_n, denoted by $\mathbb{M}_n$, is itself a Moore family (see Figure 1). Indeed, the system of all Moore families on S_n contains a maximum element (2^{S_n}) and the intersection of two Moore families is itself a Moore family. To get an overall view of the properties of this closure system, see the survey of Caspard and Monjardet [5].



$\mathcal{M} \in \mathbb{M}_3$ $\qquad$ $\mathbb{M}_2$

Fig. 1 On the left, the Hasse diagram of a Moore family $\mathcal{M}$ on U_3. On the right, the Hasse Diagram of the complete set $\mathbb{M}_2$ of Moore families on the ground set U_2. We have $|\mathbb{M}_2| = 7$.

Some researches focus on quantitative properties of this lattice of Moore families. As an example, Demetrovics *et al.* in [11] note that the problem of counting Moore families on n elements is a complex issue for which there is no known formula. In [4], Burosh *et al.* consider the issue of counting Moore families as natural and provide an upper bound. An often supported approach to try to obtain such a formula involves counting the number of objects for the first values of n. In [6], Colomb *et al.* found 10^{19} families for $n = 7$.

In this paper we give a new recursive definition of the lattice of Moore families by building the lattice $\mathbb{M}_{n+1}$ from the lattice $\mathbb{M}_n$.

2 Preliminary study

2.1 Definitions and notations

We note elements by numbers $(1, 2, 3, \dots)$. Sets are denoted by capital letters $(A, B, C, \dots)$. Families of sets are denoted by cursive letters $(\mathcal{A}, \mathcal{B}, \dots)$. Finally, we denote the sets of families of sets by black board letters $(\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots)$. Let $\mathcal{M}$ be a family on S_n, we denote $(\mathcal{M}, \subseteq)$, or directly $\mathcal{M}$ (when no confusion is possible.), the corresponding ordered set. Given a family $\mathcal{M}$, a subfamily $\mathcal{I}$ of $\mathcal{M}$ is an *ideal* of $\mathcal{M}$ if it satisfies the following implication for any pair M, M' in $\mathcal{M}$, $M \subseteq M'$ and $M' \in \mathcal{I} \Rightarrow M \in \mathcal{I}$. We shall use $\mathbb{I}_{\mathcal{M}}$ to denote the sets of ideals on $\mathcal{M}$. Given a set X in a family $\mathcal{M}$, there exists a unique ideal $\mathcal{I}$ of $\mathcal{M}$ with X as a maximum set (also called the *principal ideal* generated by X in $\mathcal{M}$). Let $\mathcal{I}_{\mathcal{M}}(X)$ denote this ideal. By extension, we denote $\mathbb{I}_{\mathbb{M}_n}(\mathcal{M})$ the principal ideal generated by $\mathcal{M}$ in the lattice $\mathbb{M}_n$. This way, $\mathbb{I}_{\mathbb{M}_n}(\mathcal{M})$ is the set of Moore sub-families of $\mathcal{M}$.

2.2 Compatible families

A Moore family $\mathcal{M}$ on S_{n+1} can be decomposed into 2 parts. The part consisting of the sets of $\mathcal{M}$ containing the element $n+1$ (denoted by $\mathcal{M}_{up}$), and the complementary part (denoted by $\mathcal{M}_{low}$). If not already present, the element S_n is inserted into $\mathcal{M}_{low}$. Naturally, $\mathcal{M} \subseteq \mathcal{M}_{up} \cup \mathcal{M}_{low}$[1]. The family $\mathcal{M}_{low}$ is clearly a family of $\mathbb{M}_n$. And the family $\mathcal{M}_{up}$ is a Moore family on S_{n+1} with the peculiarity that all its sets contain the element $n+1$. We will denote $\mathbb{M}_{n+1}^{up}$ as the set of Moore families having this property.

Example: Let $\mathcal{M}$ be the family given in Figure 1. $\mathcal{M}$ can be decomposed into $\mathcal{M}_{low} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $\mathcal{M}_{up} = \{\{3\}, \{1,3\}, \{1,2,3\}\}$.

To study the matching conditions between a family in $\mathbb{M}_n$ and a family in $\mathbb{M}_{n+1}^{up}$, we will say that a family in $\mathbb{M}_{n+1}^{up}$ is *compatible* with a family in $\mathbb{M}_n$ if the union of both families is a Moore family in $\mathbb{M}_{n+1}$. Figure 2 illustrates that for a fixed lower part, there are several *compatible* upper parts.

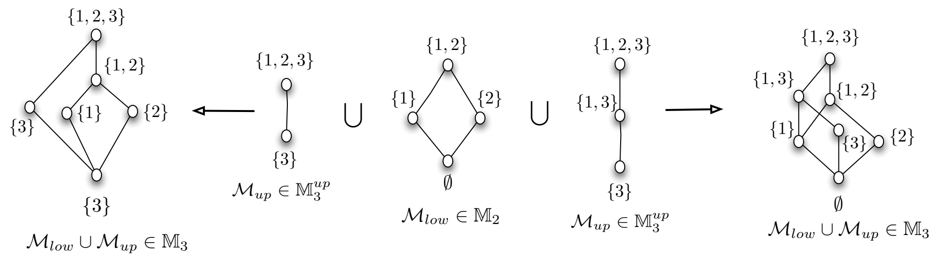


Fig. 2 In the middle, a family in $\mathbb{M}_2$ and, on each side, two different compatible families in $\mathbb{M}_3^{up}$. In both cases, the obtained family, on the opposite sides, is itself a Moore family.

2.3 Maximal compatible family

Definition 1 For any integer $n \geq 1$, we define two maps from $\mathbb{M}_n$ to $\mathbb{M}_{n+1}^{up}$:
$g_{n+1}(\mathcal{M}) = \{M \cup \{n+1\} \mid M \in \mathcal{M}\}$
$f_{n+1}(\mathcal{M}) = \{X \in 2^{S_{n+1}} \mid \{n+1\} \in X \text{ and } \forall M \in \mathcal{M} \setminus S_n,\ M \cap X \in \mathcal{M}\}$

Map g_{n+1} defines a one-to-one mapping between $\mathbb{M}_n$ and $\mathbb{M}_{n+1}^{up}$ (cf. Fig. 3,4).

Proposition 1 *Let $\mathcal{M} \in \mathbb{M}_n$, then $\forall \mathcal{M}_{up} \in \mathbb{M}_{n+1}^{up}$, the two following assertions are equivalent :*

(i) $\mathcal{M}_{up}$ is compatible with $\mathcal{M}$ (resp. $\mathcal{M} \setminus S_n$);
(ii) $\mathcal{M}_{up} \subseteq g_{n+1}(\mathcal{M})$ (resp. $\mathcal{M}_{up} \subseteq f_{n+1}(\mathcal{M})$).

[1] If S_n belongs to $\mathcal{M}$ we have $\mathcal{M} = \mathcal{M}_{up} \cup \mathcal{M}_{low}$

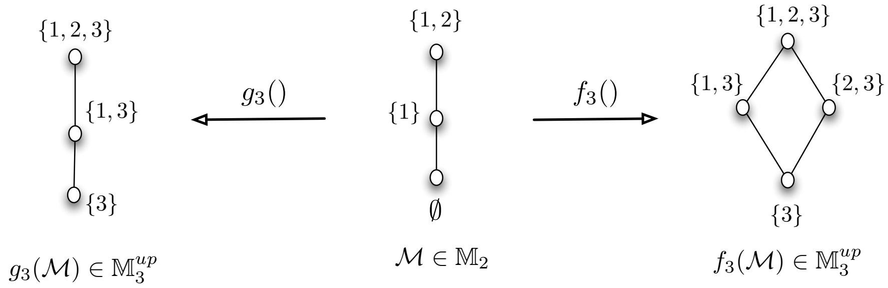


Fig. 3 In the middle, $\mathcal{M} \in \mathbb{M}_2$. On left, its image by g_3, which consists in adding the element $\{3\}$ to each set of the family. Any Moore sub-family of $g_3(\mathcal{M})$ is compatible with $\mathcal{M}$. On right, its image by f_3. Any Moore sub-family of $f_3(\mathcal{M})$ is compatible with $\mathcal{M} \setminus S_2$.

In other words, let $\mathcal{M} \in \mathbb{M}_n$, the set of compatible families with $\mathcal{M}$ (resp. $\mathcal{M} \setminus S_n$) is the set of Moore sub-families of $g_{n+1}(\mathcal{M})$ (resp. $f_{n+1}(\mathcal{M})$). This way, $g_{n+1}(\mathcal{M})$ (resp. $f_{n+1}(\mathcal{M})$) is called the *maximal compatible family* of $\mathcal{M}$ (resp. $\mathcal{M} \setminus S_n$). See Figure 4.

Corollary 1 *Let $\mathcal{M}$ be in $\mathbb{M}_n$. Then, the set of Moore families compatible with $\mathcal{M}$ (resp. $\mathcal{M} \setminus S_n$) is $\mathbb{I}_{\mathbb{M}_{n+1}^{up}}(g_{n+1}(\mathcal{M}))$ (resp. $\mathbb{I}_{\mathbb{M}_{n+1}^{up}}(f_{n+1}(\mathcal{M}))$).*

3 Recursive decomposition theorem

Definition 2 For any integer $n \geq 1$, we define the map h_n from $\mathbb{M}_n$ to $\mathbb{M}_n$: $h_n(\mathcal{M}) = g_n^{-1}(f_{n+1}(\mathcal{M}))$.

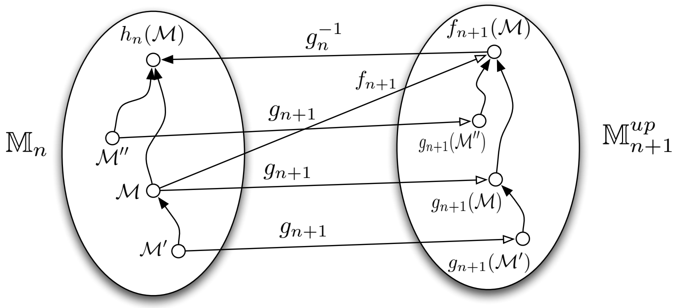


Fig. 4 On the left, the set $\mathbb{M}_n$ and on the right its isomorphic image $\mathbb{M}_{n+1}^{up}$ by g_{n+1}. Let $\mathcal{M} \in \mathbb{M}_n$, then, for any $\mathcal{M}', \mathcal{M}'' \in \mathbb{M}_n$ with $\mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{M}'' \subseteq h_n(\mathcal{M})$ we know, from Proposition 1, that $g_{n+1}(\mathcal{M}')$ (resp. $g_{n+1}(\mathcal{M}'')$) is compatible with $\mathcal{M}$ (resp. with $\mathcal{M} \setminus S_n$). This way, for any $\mathcal{M}, \mathcal{M}' \in \mathbb{M}_n$ (resp. $\mathcal{M}, \mathcal{M}''$) such that $\mathcal{M}' \subseteq \mathcal{M}$ (resp. $\mathcal{M}'' \subseteq h_n(\mathcal{M})$), the families $\mathcal{M} \cup g_{n+1}(\mathcal{M}')$ and $\mathcal{M} \setminus S_n \cup g_{n+1}(\mathcal{M}'')$ are Moore families on S_{n+1}.

Theorem 1 states that the set of Moore families on S_{n+1} can be obtained from the set of Moore families on S_n.

Theorem 1 *For any integer n such that $n \geq 1$,*

$$\mathbb{M}_{n+1} = \begin{array}{ll} \bigcup_{\mathcal{M}\in\mathbb{M}_n}\{\mathcal{M}\cup g_{n+1}(\mathcal{M}') \mid \mathcal{M}' \in \mathbb{I}_{\mathbb{M}_n}(\mathcal{M})\} & \cup \\ \bigcup_{\mathcal{M}\in\mathbb{M}_n}\{\mathcal{M}\setminus S_n \cup g_{n+1}(\mathcal{M}'') \mid \mathcal{M}'' \in \mathbb{I}_{\mathbb{M}_n}(h_n(\mathcal{M}))\}. & \end{array}$$

In other words, there exists a natural bi-partition of $\mathbb{M}_{n+1}$: the families containing the set S_n, under the form $\mathcal{M}\cup g_{n+1}(\mathcal{M}')$ with $\mathcal{M}'$ a Moore sub-family of $\mathcal{M}$, and the families not containing S_n, under the form $\mathcal{M}\setminus S_n \cup g_{n+1}(\mathcal{M}'')$ with $\mathcal{M}''$ a Moore sub-family of $h_n(\mathcal{M})$.

4 Conclusion

In this article we give a new recursive definition of the lattice of Moore families. We hope this new definition will be a useful tool to tackle some open problems on Moore families like enumeration problems or Frankl's conjecture.

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